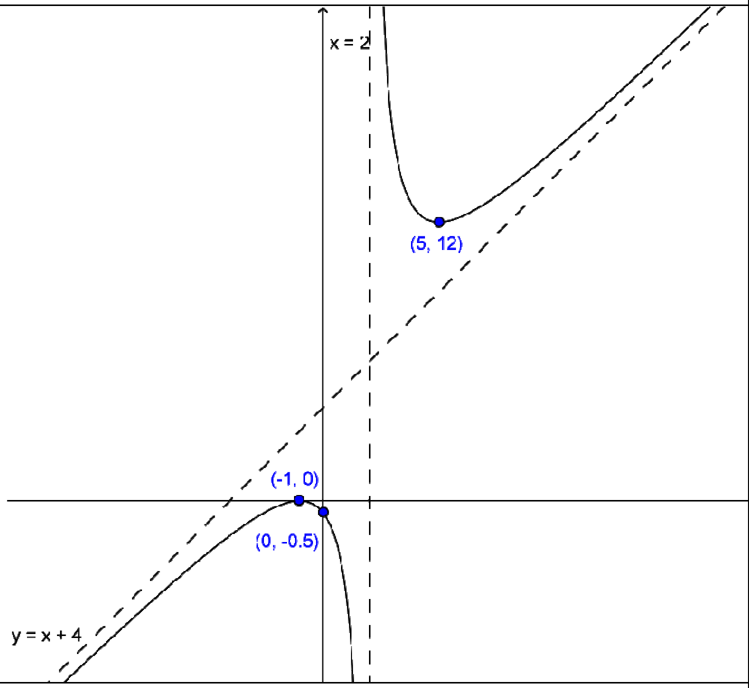
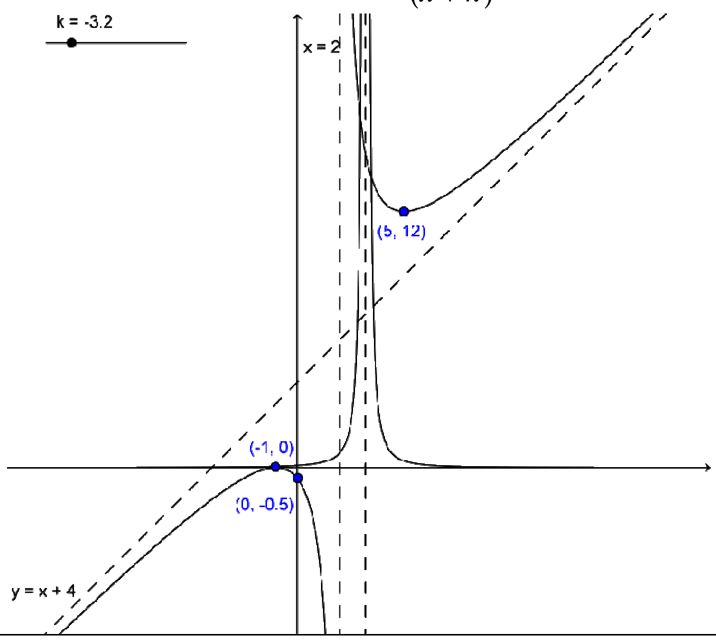
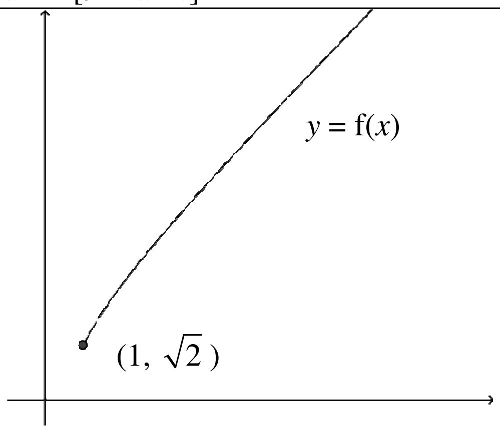
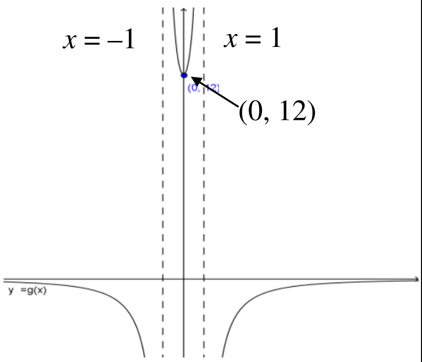
RVHS 2015 Y6 H2 MA Prelim Paper 2 (Solutions)

Section A: Pure Mathematics [40 marks]

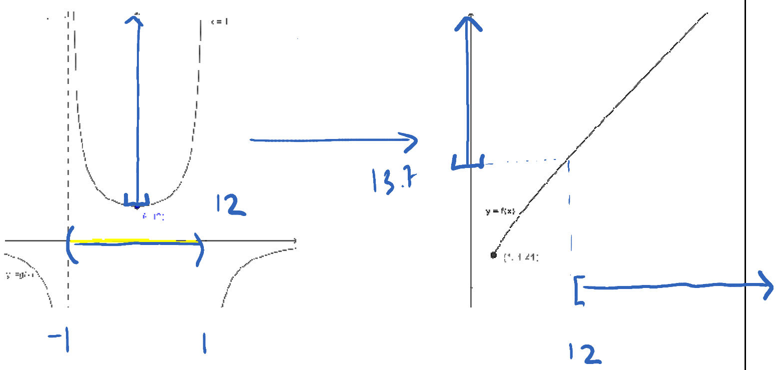
Question 1 [8 Marks]		
(i)	Vertical asymptote when $x = 2$ $2 + b = 0 \rightarrow b = -2$ Substitute $x = -1$, $y = 0$ into the equation for C: $0 = \dfrac{(-1)^2 + a(-1) + 1}{(-1) + b} \rightarrow a = 2$	
(ii)		
(iii)	$\left[(x+1)(x+k)\right]^2 = x - 2$ $\dfrac{(x+1)^2}{x-2} = \dfrac{1}{(x+k)^2}, \quad x \neq 2$ Therefore, draw the graph $y = \dfrac{1}{(x+k)^2}$ 	

	By observation, the graphs intersect twice when $k < -2$. Furthermore, when $k = -2$, equation becomes: $(x+1)^2 (x-2)^2 = x-2$ $(x-2)[(x+1)^2 (x-2)-1] = 0$ $x = 2$ or $x = 2.10380$ $\therefore k = -2$ is an answer as well. Ans: $k \leq -2,\ k \in \mathbb{Z}$	

Question 2 [9 Marks]		
(i)	 Any horizontal line $y = k$ will cut the graph of $y = \mathrm{f}(x)$ at most once. Therefore, f is one-one.	
(ii)	Let $y = \mathrm{f}(x)$. Then: $y = \sqrt{x^2 + 4x - 3}$ $y^2 = (x+2)^2 - 7$ $x = -2 \pm \sqrt{y^2 + 7}$ (Rej negative $\because x \geq 1$) Thus, $\mathrm{f} : x \mapsto -2 + \sqrt{x^2 + 7}, \quad x \geq \sqrt{2}$	
(iii)	fg exists iff $R_{\mathrm{g}} \subseteq D_{\mathrm{f}}$ i.e. $R_{\mathrm{g}} \subseteq [1, \infty)$ So, $R_{\mathrm{g}} = [12, \infty)$ $\Rightarrow D_{\mathrm{g}} = (-1, 1)$ 	

(iv)

$D_g = (-1,1) \xrightarrow{g} [12,\infty) \xrightarrow{f} [\sqrt{189},\infty)$ (OR $[13.7,\infty)$)



Therefore, $R_{fg} = [13.7,\infty)$

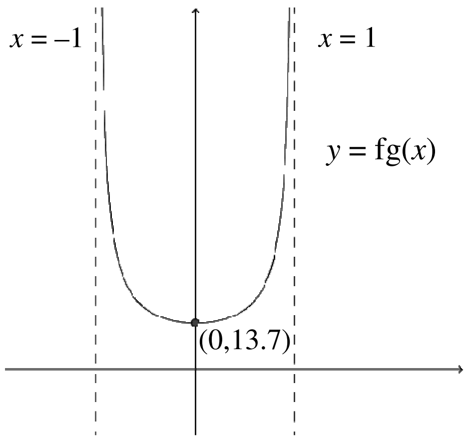
<u>Alternatively:</u>

$\text{fg}(x)$

$= \text{f}(\text{g}(x))$

$= \text{f}\left(\dfrac{12}{1-x^2}\right)$

$= \sqrt{\left(\dfrac{12}{1-x^2}\right)^2 + 4\left(\dfrac{12}{1-x^2}\right) - 3}$

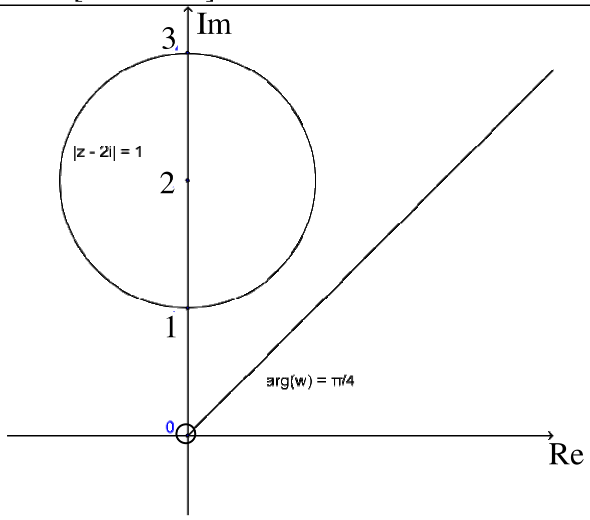


From the graph,

$R_{fg} = [13.7,\infty)$

Question 3 [11 Marks]

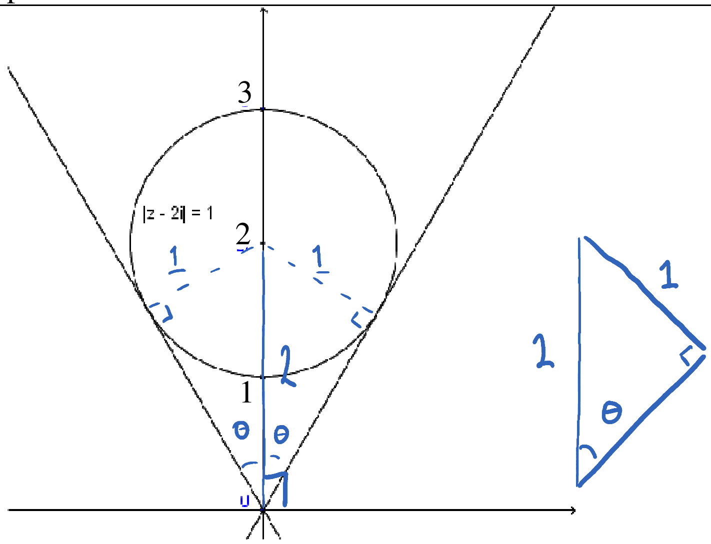
(i)



$|z-2i|=1$ is a circle centred on (0, 2) with radius 1.

$\arg(w)=\dfrac{\pi}{4}$ is the half line from the origin (excluding the origin), which makes an angle of $\dfrac{\pi}{4}$ with the positive real-axis.

(ii)

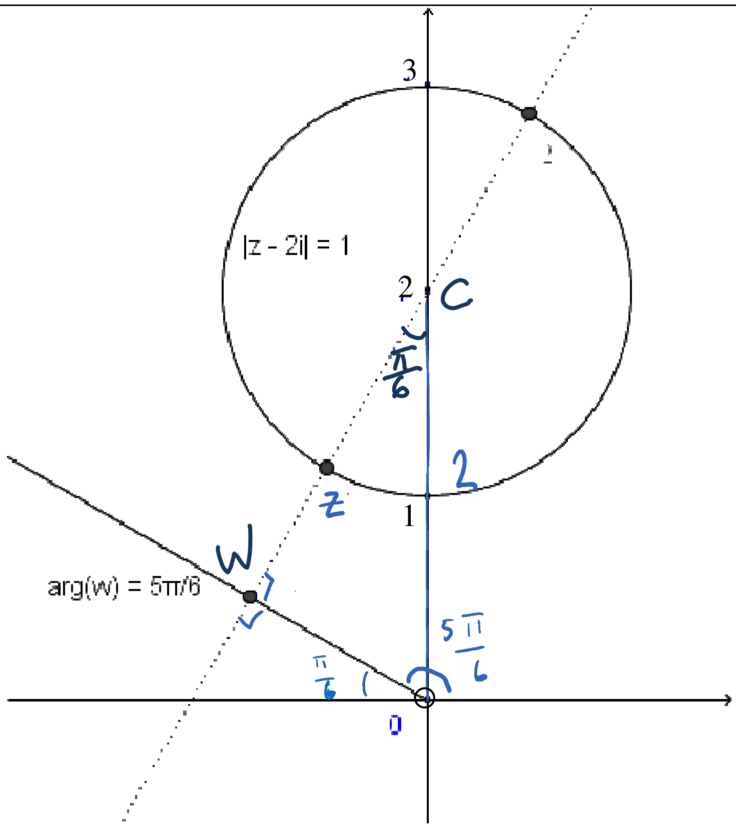


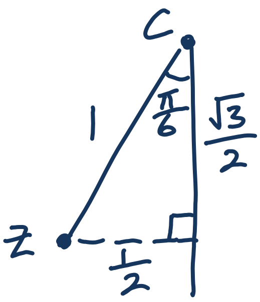
Intersect at one point implies the half line is a tangent to the circle.

$\sin\theta=\dfrac{1}{2}\Rightarrow\theta=\dfrac{\pi}{6}$

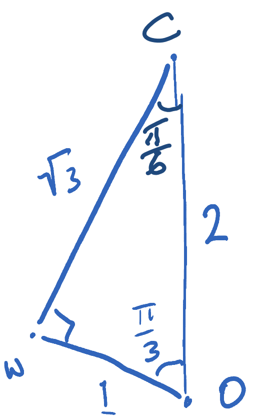
Therefore, $k=\dfrac{\pi}{2}-\dfrac{\pi}{6}=\dfrac{\pi}{3}$ or $k=\dfrac{\pi}{2}+\dfrac{\pi}{6}=\dfrac{2\pi}{3}$

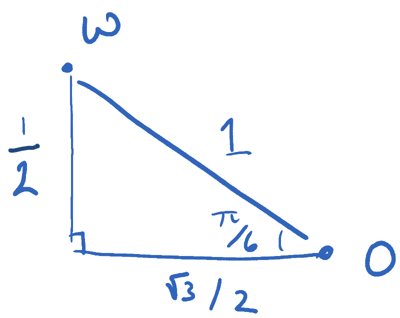
(iii)





$$z = 2i - \frac{\sqrt{3}}{2}i - \frac{1}{2} = \frac{-1}{2} + i\frac{4-\sqrt{3}}{2}$$





$$w = \frac{-\sqrt{3}}{2} + \frac{i}{2}$$

Question 4 [12 Marks]

(a)

$$y = ux + 2 \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{du}{dx} + u$$

Substituting into DE:

$$\frac{dy}{dx} - \sqrt{4-(y-2)^2} = 0$$

$$\frac{dy}{dx} - \sqrt{4-(y^2-4y+4)} = 0$$

$$\frac{dy}{dx} = \sqrt{4y-y^2} \quad \text{(shown)}$$

$$\therefore \int \frac{1}{\sqrt{4y-y^2}}\, dy = \int 1\, dx$$

$$\int \frac{1}{\sqrt{4-(y-2)^2}}\, dy = \int 1\, dx$$

$$\sin^{-1}\left(\frac{y-2}{2}\right) = x + c$$

$$y - 2 = 2\sin(x+c)$$

$$u = \frac{2\sin(x+c)}{x}$$

(b) (i)

From part (i), $s = 2\sin(t+c) + 2$

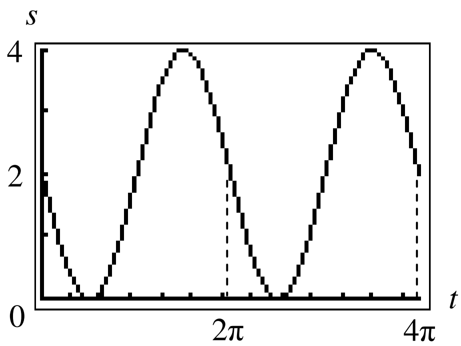
when $t = \frac{5\pi}{6}$, $s = 1$:

$$\sin\left(\frac{5\pi}{6} + c\right) = -\frac{1}{2}$$

$$\frac{5\pi}{6} + c = -\frac{\pi}{6}$$

$$c = -\pi$$

$$\therefore s = 2\sin(t-\pi) + 2$$



(ii)

The object oscillates about the starting point which is 2m from O, with an amplitude of 2m.

The motion assumes the absence of resistance whereby the amplitude remains constant which is unrealistic in real-life.

Section B: Statistics [60 marks]

Question 5 [3 Marks]		
(i)	No. of ways = $[(i+j+k)-1]!$	
(ii)	No. of ways = $3!i!j!k!=6i!j!k!$	

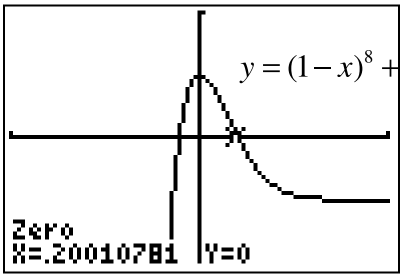
Question 6 [6 Marks]		
(a)	A ***population*** is the <u>entire collection</u> of data that we want to study while a ***sample*** is a <u>subset of</u> the units in <u>the population</u> having the same characteristics that we want to measure.	
(b) (i)	This procedure will not result in a random sample as each of the 120 teachers does not have an equal chance of being selected, e.g. P(a Humanities teacher chosen) = $\frac{10}{21}$ while P(a Language teacher chosen) = $\frac{10}{33}$.	
(ii)	We calculate the number of teachers from each department based on simple ratio and proportion:	

Department	**Number of teachers selected**
Humanities	$\frac{21}{120}\times 40=7$
Language	$\frac{33}{120}\times 40=11$
Mathematics	$\frac{27}{120}\times 40=9$
Science	$\frac{39}{120}\times 40=13$

7 Humanities teachers, 11 Language teachers, 9 Mathematics teachers and 13 Science teachers are chosen randomly from each department.

<table>
<tr><td colspan="3">Question 7 [7 Marks]</td></tr>
<tr><td></td><td>Let X denote the number of correct answers gotten by someone totally clueless of Singapore's history.
Then $X \sim B(8, p)$, where p is the probability of getting a correct answer by mere guessing.

Given $P(X \le 1) = 0.50331648$
$\Rightarrow (1-p)^8 + \binom{8}{1} p(1-p)^7 = 0.503$
$(1-p)^8 + 8p(1-p)^7 - 0.503 = 0$



From the graph, $p \approx 0.200 = \frac{1}{5}$.
Therefore, there are 5 options for each of the question. (shown)</td><td></td></tr>
<tr><td></td><td>So $X \sim B(8, 0.2)$
$P(X = 0) = 0.1678$
$P(X = 1) = 0.3355$
$P(X = 2) = 0.2936$
The mode is 1.</td><td></td></tr>
<tr><td></td><td>$\overline{X} \sim N\left(1.6, \frac{1.28}{50}\right)$ approx. by CLT, since n = 50 is large.
$\therefore P(\overline{X} \ge 2) = 0.00621$ (to 3 s.f.)</td><td></td></tr>
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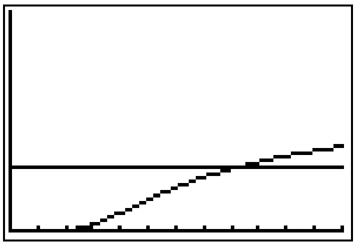
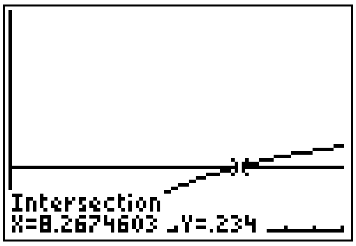
<table>
<tr><td colspan="3">Question 8 [7 Marks]</td></tr>
<tr><td>(i)</td><td><u>The average number of calls</u> requiring ambulance assistance <u>is constant throughout</u> the operating hours of the hospital in <u>a day</u>.
OR
<u>For any time interval within</u> the operating hours of the hospital in <u>a day</u>, <u>the mean number of calls</u> requiring ambulance <u>assistance is proportional to the time interval</u>.</td><td></td></tr>
<tr><td>(ii)</td><td>Let X denote the number of calls requiring ambulance assistance in 1 week.
Then $X \sim Po(7)$.

Required probability
$= P(X < 5) = P(X \le 4) = 0.172992 = 0.173$ (to 3sf)</td><td></td></tr>
</table>

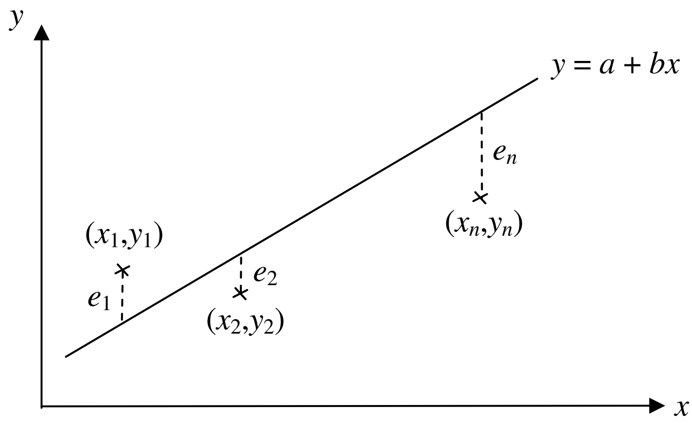
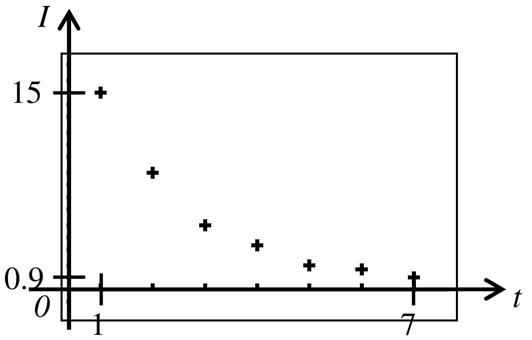
(iii)	Let Y denote the number of weeks (out of 52) with more than 4 calls requiring ambulance assistance. Then $Y \sim B(52, 1-0.172992)$ i.e. $Y \sim B(52, 0.827008)$ Since $n = 52$ is large, $np = 43.004416 > 5$ and $nq = 8.995584 > 5$, $\therefore Y \sim N(43.004416, 7.439419933)$ approx. Required probability $= P(40 < Y \leq 45)$ $= P(40.5 < Y < 45.5)$ C.C. $= 0.641$ (to 3sf)	

Question 9 [8 Marks]		
(i)	Let X denote the time required by the machine to complete a task and μ be the mean time. Test $H_0 : \mu = 47.0$ against $H_1 : \mu > 47.0$	
(ii)	$\bar{x} = 48.1$ $s^2 = \frac{n}{n-1}\sigma_n{}^2 = \frac{12}{11}(1.9)^2 = 3.938181818$ Test statistic: Under H_0, $T = \frac{\bar{X} - 47.0}{\sqrt{\frac{3.938181818}{12}}} \sim t(11)$ Using GC, p-value = 0.0405659745 > 0.03 (level of significance) $\therefore$ do not rej. H_0. There is insufficient evidence at 3% level of significance that the mean time required by the machine to complete the task is understated.	
(iii)	Given: $\sigma = 2.1$ Test statistic: Under H_0, $Z = \frac{\bar{X} - 47.0}{2.1/\sqrt{12}} \sim N(0,1)$ For H_0 to be rejected, $Z_{cal} > Z_{critical}$ i.e. $\frac{\bar{x} - 47.0}{2.1/\sqrt{12}} > 1.88079361$ $\Rightarrow \bar{x} > 48.14017053$ $\therefore \bar{x} > 48.1$ (or $\bar{x} \geq 48.2$)	

Question 10 [8 Marks]		
(i)	P(all 3 balls have different colours) $= {}^4P_3 \times \left(\frac{1}{4}\right)^3$ $= \frac{3}{8}$ or 0.375	
(ii)	$\frac{\text{P(the 3 balls have diff. colours and nos '0,0,0' or '0,1,1')}}{\text{P(all 3 balls have different colours)}}$ $= \frac{{}^4P_3 \times \left(\frac{1}{8}\right)^3 + {}^4P_3 \times {}^3C_2 \times \left(\frac{1}{8}\right)^3}{\frac{3}{8}}$ $= \frac{1}{2}$	
	$\mathrm{P}(A \mid B) = \frac{1}{2}$ (from part (ii)) $\mathrm{P}(A) = \mathrm{P}(0,0,0) + \mathrm{P}(0,1,1)$ $= \left(\frac{1}{2}\right)^3 + {}^3C_1 \times \left(\frac{1}{2}\right)^3$ $= \frac{1}{2} = \mathrm{P}(A \mid B)$ $\therefore$ A and B are independent. <u>ALTERNATIVE METHOD</u> $\mathrm{P}(B) = \frac{3}{8}$ (from part (i)) $\mathrm{P}(A) = \mathrm{P}(0,0,0) + \mathrm{P}(0,1,1)$ $= \left(\frac{1}{2}\right)^3 + {}^3C_1 \times \left(\frac{1}{2}\right)^3$ $= \frac{1}{2}$ $\therefore \mathrm{P}(A) \cdot \mathrm{P}(B) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$ In addition, $\mathrm{P}(A \cap B) = \frac{3}{16}$ (from (ii)) $= \mathrm{P}(A) \cdot \mathrm{P}(B)$ Hence, A and B are independent.	

Question 11 [9 Marks]		
(i)	Let A and B denote the masses of a random mobile phone sold by Company A and B respectively. $B \sim \mathrm{N}\left(\mu,\ \sigma^2\right)$ Given $\mathrm{P}(B<134)=\mathrm{P}(B>146)$ By symmetry, $\mu=\dfrac{134+146}{2}=140$ <u>Method 1</u> $\mathrm{P}(B<134)=0.234$ $\mathrm{P}\left(Z<\dfrac{134-140}{\sigma}\right)=0.234$ $\dfrac{-6}{\sigma}=-0.7257370278$ $\sigma=8.267457453\approx 8.27$ ——— <u>Method 2</u> Sketch graph of $y=normalcdf\,(-1E99,134,140,x)$ and $y=0.234$, and find intersection:  ——— <u>Method 3</u> $\mathrm{P}(B>146)=0.234$ $\mathrm{P}\left(Z<\dfrac{146-140}{\sigma}\right)=0.766$ $\dfrac{6}{\sigma}=0.7257370278$ $\sigma=8.267457453\approx 8.27$	
(ii)	$A \sim \mathrm{N}(130,6^2)$ $\qquad B \sim \mathrm{N}(136,8^2)$ $2B \sim \mathrm{N}(272,256)$ $A-2B \sim \mathrm{N}(-142,292)$ $\therefore \mathrm{P}\left(\lvert A-2B\rvert<150\right)=\mathrm{P}\left(-150<A-2B<150\right)$ $=0.6801665426$ ≈ 0.680	

(iii)	Required probability $= \mathrm{P}(A_1 \le 135) \times \mathrm{P}(A_2 \le 135) \times \ldots \times \mathrm{P}(A_{10} \le 135)$ $= [\mathrm{P}(A \le 135)]^{10}$ $= 0.1042897699$ ≈ 0.104	
(iv)	The masses of the mobile phones sold by Company *A* are all independent of each other.	

Question 12 [12 Marks]		
(a)	Let the sample of bivariate data be (x_i, y_i) where $i = 1, 2, \ldots n$. Let $e_i = y_i - (a + bx_i)$ be the vertical deviation between the point (x_i, y_i) and the line $y = a + bx$.  The line $y = a + bx$ is the least square regression line for the sample of bivariate data if the sum of the squares of the vertical deviations, i.e. $\sum_{i=1}^{n} (e_i)^2$, is the minimum.	
(b) (i)	 	
(ii)	$r = -0.901356207 \approx -0.901$ Although $\lvert r \rvert$ is close to 1, suggesting a linear relationship between *I* and *t*, from the scatter diagram, it seems that a curvilinear relationship is a better fit.	

(iii)	We can determine which of these 2 formulae is a better model by calculating the r-value of each of these 2 formulae and then choose the formula whose $\lvert r \rvert$ is closer to 1.	
(iv)	For $I = c + dt^2$, $r = -0.7940683332 \approx -0.794$ For $I = e + f \ln t$, $r = -0.9838669707 \approx -0.984$ Hence, $I = 14.06964212 - 7.341502661 \ln t$ is used. When $I = 2.5$, $t = 4.835201938 \approx 4.84$ The estimate is reliable because: • $I = 2.5$ is within the data range provided • $\lvert r \rvert \approx 0.984$ is close to 1 suggesting a strong linear correlation • The appropriate regression line (*I* on *t*) is used, since *t* is the independent/ fixed variable	