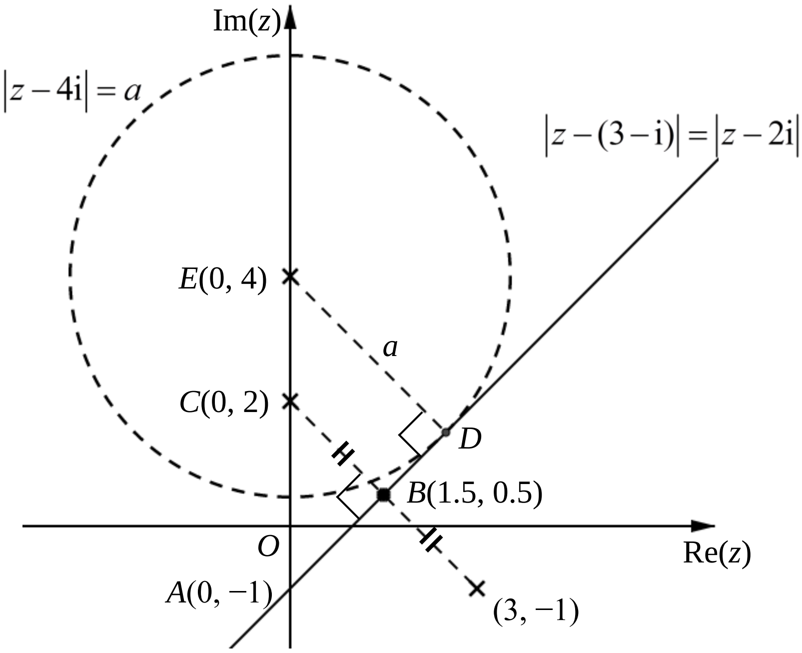
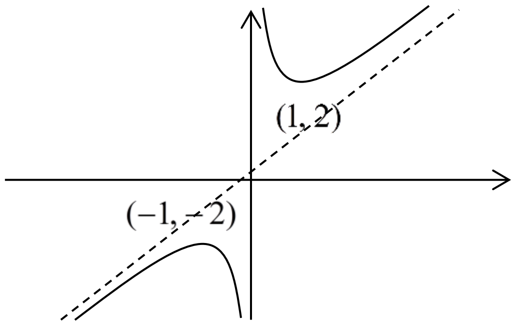
2015 ACJC H2 Mathematics Prelim Paper 2 solutions

1(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2x^2+1}\right)=\frac{-4x}{\left(2x^2+1\right)^2}.$
1(ii)	$\int\frac{x^2}{\left(2x^2+1\right)^2}\ \mathrm{d}x=\frac{-1}{4}\int x\left[\frac{-4x}{\left(2x^2+1\right)^2}\right]\mathrm{d}x$ $=\frac{-x}{4\left(2x^2+1\right)}-\left(\frac{-1}{4}\right)\int(1)\left[\frac{1}{2x^2+1}\right]\mathrm{d}x$ $=\frac{-x}{4\left(2x^2+1\right)}+\frac{1}{8}\int\frac{1}{x^2+\left(\frac{1}{\sqrt{2}}\right)^2}\ \mathrm{d}x$ $=\frac{-x}{4\left(2x^2+1\right)}+\frac{\sqrt{2}}{8}\tan^{-1}\left(x\sqrt{2}\right)+c.$
1(iii)	$x=2y,\ y=\frac{x}{2x^2+1},\ x\geq 0$ $\Rightarrow x=2y, y=\frac{2y}{8y^2+1},\ x\geq 0$ $\Rightarrow y=0, x=0\quad \text{or}\quad y=\frac{1}{2\sqrt{2}}, x=\frac{1}{\sqrt{2}}.$ The line intersects C at $(0,0)$ and $\left(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}}\right).$ $\pi\int_0^{\frac{1}{\sqrt{2}}}\left(\frac{x}{2x^2+1}\right)^2-\left(\frac{x}{2}\right)^2\ \mathrm{d}x$ $=\pi\left[\frac{-x}{4\left(2x^2+1\right)}+\frac{\sqrt{2}}{8}\tan^{-1}\left(x\sqrt{2}\right)-\frac{x^3}{12}\right]_0^{\frac{1}{\sqrt{2}}}$ $=\pi\left(\frac{-1}{8\sqrt{2}}+\frac{\pi\sqrt{2}}{32}-\frac{1}{24\sqrt{2}}\right)$ $=\pi\left(\frac{\pi\sqrt{2}}{32}-\frac{\sqrt{2}}{12}\right)$ $=\frac{\pi\sqrt{2}}{4}\left(\frac{\pi}{8}-\frac{1}{3}\right)\text{unit}^3.$
2(i)	$\left\lvert\frac{z+\mathrm{i}-3}{2+\mathrm{i}z}\right\rvert=1$ Let $z=k\mathrm{i}$ where $k\in\mathbb{R}$ $\lvert k\mathrm{i}+\mathrm{i}-3\rvert=\lvert 2+\mathrm{i}(k\mathrm{i})\rvert$ $\lvert -3+(k+1)\mathrm{i}\rvert^2=\lvert 2-k\rvert^2$ $(-3)^2+(k+1)^2=(2-k)^2$

	$9+k^2+2k+1=4-4k+k^2$ $6k=-6$ $k=-1$
2(ii)	$\lvert z-3+\mathrm{i}\rvert=\lvert -\mathrm{i}\rvert\lvert 2+\mathrm{i}z\rvert$ $\lvert z-(3-\mathrm{i})\rvert=\lvert z-2\mathrm{i}\rvert$ 
2(iii)	The locus of points representing w such that $\lvert w-4\mathrm{i}\rvert=a$ is a circle with centre (0, 4) and radius a units. To have exactly one value of z satisfying the 2 conditions, the perpendicular bisector should be tangent to the circle (see sketch in **2(ii)**) Gradient of $\perp$ bisector $=-1\div\dfrac{-1-2}{3-0}$ $=1$ $\Rightarrow$ $\perp$ bisector makes an angle of $\frac{\pi}{4}$ rad. with the horizontal axis Consider right-angled triangle *ADE*: $\sin\dfrac{\pi}{4}=\dfrac{a}{4-(-1)}$ $a=\dfrac{5}{\sqrt{2}}$ or $\dfrac{5\sqrt{2}}{2}$ $z=\dfrac{5}{\sqrt{2}}\cos\dfrac{\pi}{4}+\mathrm{i}\left(4-\dfrac{5}{\sqrt{2}}\sin\dfrac{\pi}{4}\right)$ $=\dfrac{5}{2}+\dfrac{3}{2}\mathrm{i}$ $\therefore\ x=\dfrac{5}{2},\ y=\dfrac{3}{2}$ **Alternative method for finding *a*:**

	Triangles ABC and ADE are similar right-angled triangles with common angle $\angle BAC$ $\sin\angle BAC=\frac{BC}{AC}=\frac{DE}{AE}$ $\frac{\sqrt{(0-1.5)^2+(2-0.5)^2}}{3}=\frac{a}{5}$ $a=\frac{5}{3}\times\sqrt{\frac{9}{2}}$ $=\frac{5}{\sqrt{2}}$ or $\frac{5\sqrt{2}}{2}$
3(i)	$y=x+\frac{1}{x}$  Asymptotes: $y=x$, $x=0$
3(ii)	$y=\mathrm{f}(x)$ is a translation of $y=x+\frac{1}{x}$ a units in the positive x-direction. Therefore the turning points are $(a+1,\,2)$ and $(a-1,-\,2)$. Hence $b=a+1$.
3(iii)	$\mathrm{R_f}=[2,\infty)\subseteq(0,1)\cup(1,\infty)=\mathrm{D_g}$ Hence gf exists. $\mathrm{R_{gf}}=\left(0,\tfrac{1}{\ln 2}\right]$
3(iv)	$\mathrm{f}(x)=x-2+\frac{1}{x-2},\ x\geq 3$ $y=x-2+\frac{1}{x-2},\ x\geq 3$ $=\frac{(x-2)^2+1}{x-2}$ $(x-2)y=x^2-4x+5$ $x^2-(4+y)x+5+2y=0$ $x=\frac{4+y\pm\sqrt{(4+y)^2-4(1)(5+2y)}}{2}=\frac{4+y\pm\sqrt{y^2-4}}{2}$ $\because x\geq 3,\ x=\frac{4+y+\sqrt{y^2-4}}{2}$ $\mathrm{f}^{-1}:x\mapsto\frac{4+x+\sqrt{x^2-4}}{2},\ x\geq 2$
4(a)	**Proving parallel & distinct planes** Normal vectors of p_1 and p_2 are respectively

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \qquad \mathbf{n}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -6 \end{pmatrix}$$

Since $\mathbf{n}_1 = -\mathbf{n}_2$, the normal vectors are parallel and hence the planes are parallel as well.
Furthermore, $3(1) - 2(-2) + 6(0) = 7 \neq 2$, so the point (1, −2, 0) is on p_2 but not on p_1.
Hence the two planes are distinct.

<u>Finding shortest distance</u>
Method 1:
Since $\frac{3}{2}(0) - (-1) + 3(0) = 1$, the point $A(0,-1,0)$ is on p_1.
The point $B(1,-2,0)$ is on p_2.

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Shortest distance between planes} &= \left|\overrightarrow{AB} \,\boldsymbol{.}\, \hat{\mathbf{n}}_1\right| \\ &= \frac{1}{\sqrt{9+4+36}} \left| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} . \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \right| \\ &= \frac{|3+2+0|}{7} \\ &= \frac{5}{7} \end{aligned}$$

Method 2:

Express equations of both planes in scalar product form $\mathbf{r.n} = d$ where $\mathbf{n}$ is a unit normal vector and d is the shortest distance between the origin and the plane.

$$p_1:\ \mathbf{r.}\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} = 2 \quad \Rightarrow \quad \mathbf{r.}\frac{1}{\sqrt{9+4+36}}\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} = \frac{2}{\sqrt{9+4+36}}$$

$$\Rightarrow \quad \mathbf{r.n} = \frac{2}{7} \text{ where } \mathbf{n} = \frac{1}{7}\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$$

$$p_2:\ \mathbf{r.}\begin{pmatrix} -3 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \boldsymbol{.} \begin{pmatrix} -3 \\ 2 \\ -6 \end{pmatrix} = -7$$

$$\Rightarrow \quad \mathbf{r.}\frac{-1}{\sqrt{9+4+36}}\begin{pmatrix} -3 \\ 2 \\ -6 \end{pmatrix} = \frac{(-1)(-7)}{\sqrt{9+4+36}}$$

$$\Rightarrow \quad \mathbf{r.n} = 1 \text{ where } \mathbf{n} = \frac{1}{7}\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$$

Since both planes are on the same side of the origin,
shortest distance between both planes $= 1 - \dfrac{2}{7}$

	$=\frac{5}{7}$
4(b)(i)	$l\text{: } r=\begin{pmatrix}5\\-5\\\alpha\end{pmatrix}+\lambda\begin{pmatrix}\beta\\3\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ Since the planes are parallel, for *l* to intersect one but not the other, we need *l* to be contained in exactly one plane at one time. $p_1\text{: } \mathbf{r.}\begin{pmatrix}3\\-2\\6\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=2$ $p_2\text{: } \mathbf{r.}\begin{pmatrix}3\\-2\\6\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=7$ *l* parallel to p_1 and p_2 $\Rightarrow\begin{pmatrix}\beta\\3\\1\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=0$ $\Rightarrow\ \beta=0$ *l* contained in p_1 $\Rightarrow\begin{pmatrix}5\\-5\\\alpha\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=2$ $\Rightarrow\ \alpha=\frac{2-15-10}{6}$ $\Rightarrow\ \alpha=-\frac{23}{6}$ *l* contained in p_2 $\Rightarrow\begin{pmatrix}5\\-5\\\alpha\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=7$ $\Rightarrow\ \alpha=\frac{7-15-10}{6}$ $\Rightarrow\ \alpha=-3$ $\therefore\ \beta=0$, and $\alpha=-\frac{23}{6}$ or -3
4(b)(ii)	$\sin 22^\circ=\left\lvert\frac{\begin{pmatrix}\beta\\3\\1\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}}{\sqrt{\beta^2+9+1}\times\sqrt{9+4+36}}\right\rvert$ $0.37461=\frac{3\lvert\beta\rvert}{7\sqrt{\beta^2+10}}$ Using GC, $\beta=-5.69$ or 5.69 (3 s.f.)
5	Number of serious delays is not modelled by a Poisson distribution since the mean is not equal to the variance. Let *X* be the random variable denoting the number of serious delays per week.

E(X) = 4.3, Var(X) = 2.56

Since $n = 60$ is large, by Central Limit Theorem,

$$\overline{X} \sim \mathrm{N}\left(4.3, \frac{2.56}{60}\right) \text{ approximately}$$

$P\left(\overline{X} \le 4\right) = 0.0732$ (3 s.f.)

Alternative Method:

Let $W = X_1 + X_2 + \cdots + X_{60}$.

Since $n = 60$ is large, by Central Limit Theorem,

$W \sim \mathrm{N}(258, 153.6)$

$P(W \le 240) = 0.0732 \ (3 \text{ s.f.})$

6(i)

Quota sampling.

Disadvantage:

The 20 boys seated on the last row of LT may belong to the same class OR

the 20 boys seated on the last row of LT may be latecomers so they may have missed part of the talk OR

Since the sampling frame is available, it is more appropriate to use simple random sampling or systematic sampling.

6(ii)

Stratified Sampling

	SA (LT1)	SB (LT2)	SC (LT3)	SD (LT4)
Number of Year 2 students selected in each LT	26	24	30	20

Select the number of students randomly from each strata (i.e. subject combination) according to the table shown above.

7(a)

$n! \ge 25$

$\Rightarrow \ n \ge 5$

7(b)(i)

<u>Method 1</u>

Case 1 (all distinct): ${}^4P_3 = 24$

Case 2 (all the same): ${}^3C_1 = 3$

Case 3 (AAB in any order): ${}^3C_1 \cdot {}^3C_2 \cdot \dfrac{3!}{2!} = 27$

Total = 24 + 3 + 27 = 54 ways

<u>Method 2</u>

Case 1 (nobody gets hazelnut): $3 \times 3 \times 3 = 27$

Case 2 (someone gets hazelnut): ${}^3C_1 \times 3 \times 3 = 27$

Total = 27 + 27 = 54 ways

7(b)(ii)	**<u>Method 1</u>** No. of ways the particular friend gets the hazelnut bar = ${}^3C_2 \cdot 2! + {}^3C_1 = 9$ (the other two friends get two distinct bars or two of the same bars) Hence no. of ways that friend DOES NOT get the hazelnut bar = total no. of ways without restriction – 9 = 54 – 9 = 45.
	<u>Method 2</u> Case 1 (nobody gets hazelnut): $3 \times 3 \times 3 = 27$ Case 2 (someone gets hazelnut): ${}^2C_1 \times 3 \times 3 = 18$ Total = 27 + 18 = 45 ways
8(i)	Let X be the capacity of a randomly chosen paper cup. Then $X \sim \mathrm{N}\left(500, 45^2\right)$. $X_1 + X_2 + \ldots + X_{21} \sim \mathrm{N}\left(21 \times 500, 21 \times 45^2\right) = \mathrm{N}\left(10500, 42525\right)$. $\mathrm{P}\left(X_1 + X_2 + \ldots + X_{21} \geq 10000\right) = 0.992$ (3 s.f.).
8(ii)	<u>Method 1:</u> $21X \sim \mathrm{N}\left(10500, \left(21 \times 45\right)^2\right) = \mathrm{N}\left(10500, 945^2\right)$ $\mathrm{P}\left(21X \geq 10000\right) = 0.70163$ (shown) <u>Method 2:</u> $\mathrm{P}\left(21X \geq 10000\right) = \mathrm{P}\left(X \geq \frac{10000}{21}\right) = 0.70163$ (shown)
8(iii)	Let Y be the number of tanks, out of 60, that will be filled. Then $Y \sim \mathrm{B}\left(60, 0.70163\right)$. For approximation, check that $np = 60 \times 0.70163 = 42.0978 > 5$ $nq = 60 \times \left(1 - 0.70163\right) = 17.9022 > 5$ Therefore, $Y \sim \mathrm{N}\left(42.0978, 12.5607\right)$ approximately. $\mathrm{P}\left(Y \leq 45\right) \overset{\text{c.c.}}{\approx} \mathrm{P}\left(Y \leq 45.5\right)$ $= 0.831$ (3 s.f.)
9(a)	From GC, Unbiased estimate of population mean, $\bar{x} = 171.6875$ Unbiased estimate of population variance, $s^2 = 2.7121^2$ Let X be the random variable for the height of a randomly chosen Taz 17-year old boy. Let μ be the population mean height of Taz 17-year old boys. To test $H_o : \mu = 170$ Against $H_1 : \mu > 170$ at 5% level of significance

	Under H_o, $T=\frac{\bar{X}-170}{\frac{s}{\sqrt{8}}}\sim t(7)$ Value of test statistic : $t=\frac{171.6875-170}{\frac{2.7121}{\sqrt{8}}}=1.76$ p-value = 0.0609 > 0.05 Do not reject H_o. There is insufficient evidence at 5% level of significance that the mean height of Taz 17-year old boys is more than 170cm i.e. There is insufficient evidence that John's belief is justified at 5% level of significance.
9(b)(i)	To test $H_o:\mu=170$ Against $H_1:\mu<170$ at 5% level of significance
9(b)(ii)	Under H_o, $\bar{X}\sim \mathrm{N}\left(170,\frac{4.2}{\sqrt{25}}\right)$ Test statistic, $Z=\frac{\bar{X}-170}{\frac{4.2}{\sqrt{n}}}\sim N(0,1)$ Value of test statistic : $z=\frac{168.6-170}{\frac{4.2}{\sqrt{n}}}$ For H_o not to be rejected at 5% level of significance $\frac{168.6-170}{\frac{4.2}{\sqrt{n}}}>-1.64485$ $n<24.4$ Thus range of values of n is $\{1\le n\le 24, n\in\mathbb{Z}^+\}$ The assumption that the heights are normally distributed is necessary in order that $\bar{X}$ will be normally distributed. This is because $\bar{X}$ will not be approximately normal by Central Limit Theorem since n is small.
10(i)	The mean number of chocolate chips in one cookie is a constant. AND Either: The chocolate chips are randomly distributed such that the occurance of chocolate chips in a cookie is independent of each other, OR The chocolate chips are randomly distributed such that the number of chocolate chips in one cookie is independent of the number in another cookie.
10(ii)	Let C be the number of chocolate chips in one cookie. Then $C\sim \mathrm{P_o}(4)$. Required probability = $\left[\mathrm{P}(C=5)\right]^2\cdot\left[\mathrm{P}(C>5)\right]^2\cdot\frac{4!}{2!2!}$ = 0.00677 (3 s.f.)
10(iii)	$\mathrm{P}(C\le 1)=0.091578$ Let X be the number of cookies in one box with at most one chocolate chip.

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Then $X \sim \mathrm{B}(30, 0.091578)$.

For approximation, check that
$np = 30 \times 0.091578 = 2.74734 < 5$.

Hence $X \sim \mathrm{P_o}(2.74734)$ approximately.

$$\begin{aligned}\mathrm{P}(X \geq 2) &= 1 - \mathrm{P}(X \leq 1)\\ &\approx 0.75980\\ &= 0.760 \text{ (3 s.f.)}\end{aligned}$$

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10(iv)

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Now $C \sim \mathrm{P_o}\left(\frac{600+n}{150}\right)$.

Given: $\mathrm{P}(C \geq 2) \geq 0.95$

$\Rightarrow \quad 1 - \mathrm{P}(C \leq 1) \geq 0.95$

$\Rightarrow \quad \mathrm{P}(C \leq 1) \leq 0.05$

$\Rightarrow \quad e^{-\frac{600+n}{150}}\left(1 + \frac{600+n}{150}\right) \leq 0.05$

From G.C., $n \geq 111.58$
Hence least integer n is 112.

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11(a)

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Method 1

Given: $\mathrm{P}(M|L) = \frac{4}{7}$.

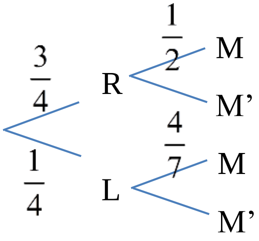
$\Rightarrow \quad \frac{\mathrm{P}(M \cap L)}{\mathrm{P}(L)} = \frac{4}{7}$

$\Rightarrow \quad \mathrm{P}(M \cap L) = \frac{4}{7} \times \mathrm{P}(L)$

$\Rightarrow \quad \mathrm{P}(M \cap L) = \frac{4}{7} \times \frac{1}{4} = \frac{1}{7}$.

Hence $\mathrm{P}(M' \cap L) = \mathrm{P}(L) - \mathrm{P}(M \cap L) = \frac{1}{4} - \frac{1}{7} = \frac{3}{28}$.

Method 2 (Using tree diagram drawn from next part of information)



$\mathrm{P}(M' \cap L) = \mathrm{P}(L) \text{ x } \mathrm{P}(M'|L) = \frac{1}{4} \times \frac{3}{7} = \frac{3}{28}$.

To determine whether being right-handed and taking mathematics are independent events

$\mathrm{P}(M) = \frac{3}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{4}{7} = \frac{29}{56}$, $\mathrm{P}(M|R) = \frac{1}{2}$

Since $\mathrm{P}(M|R) \neq \mathrm{P}(M)$, being right-handed and taking mathematics are NOT independent events.
ALTERNATIVELY,

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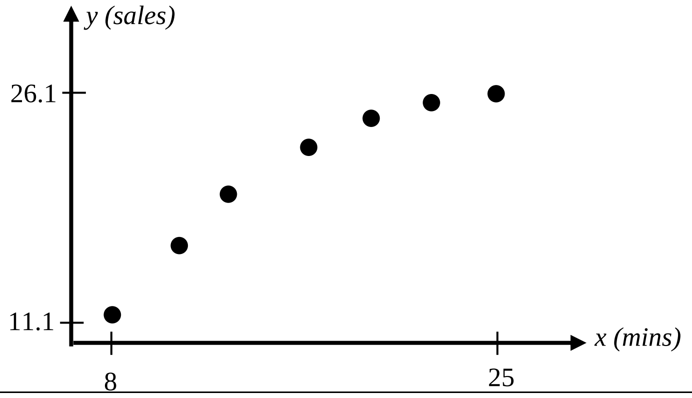
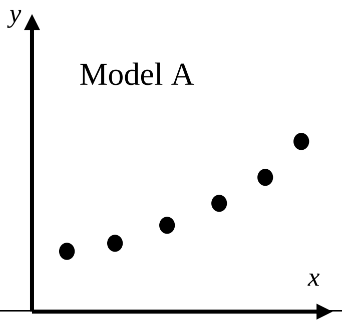
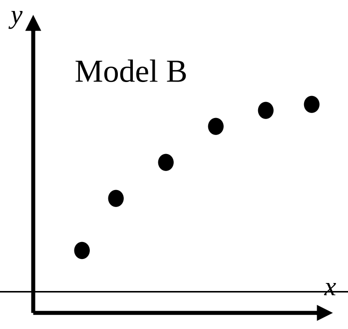
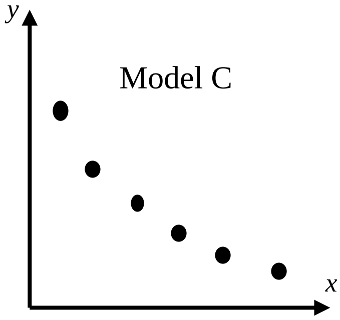
	Since it is given that $\mathrm{P}\left(M \mid L\right)=\frac{4}{7}\neq\frac{29}{56}=\mathrm{P}\left(M\right)$, being left-handed and taking mathematics are NOT independent events. Thus, being right-handed and taking mathematics are NOT independent events. ALTERNATIVELY, $\mathrm{P}(M)=\frac{3}{4}\times\frac{1}{2}+\frac{1}{4}\times\frac{4}{7}=\frac{29}{56}$ $\mathrm{P}(M\cap R)=\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}$, $\mathrm{P}(M)\times\mathrm{P}(R)=\frac{29}{56}\times\frac{3}{4}=\frac{87}{224}$ Since $\mathrm{P}(M\cap R)\neq\mathrm{P}(M)\times\mathrm{P}(R)$, being right-handed and taking mathematics are NOT independent events.
11(b)(i)	Required probability = P (ace, picture) $=\frac{4}{52}\times\frac{12}{51}$ $=\frac{4}{221}$ or 0.0181 (3 s.f.)
11(b)(ii)	Required probability = P (picture) + P (ace, picture) + P (ace, ace, picture) + P (ace, ace, ace, picture) + P (ace, ace, ace, ace, picture) $=\frac{12}{52}+\frac{4}{52}\cdot\frac{12}{51}+\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{12}{50}+\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\cdot\frac{12}{49}+\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\cdot\frac{1}{49}\cdot\frac{12}{48}$ $=\frac{1}{4}$.
11(b)(iii)	Required probability = $\frac{\text{P (wins on first card picked)}}{\text{P (player wins the game)}}$ $=\frac{12}{52}\div\frac{1}{4}$ $=\frac{12}{13}$ or 0.923 (3 s.f.)
12(i)	
12(ii)	 

	 The scatter diagram in (i) shows that as x increases, y increases by decreasing amounts (or at a decreasing rate). This is consistent with model B. Furthermore, $r = 0.997$ for Model B, which suggests a strong positive linear relationship between ln x and y. Model A is not appropriate as it suggests that as x increases, y increases by increasing amounts (or at an increasing rate). Model C is not appropriate since it suggests that as x increases, y decreases, which is not consistent with the scatter diagram in (i). Equation of least square regression line of y on ln x is $y = –17.326 + 13.698 \ln x$ $a = –17.3$ (3 s.f.) , $b = 13.7$ (3 s.f.) $r = 0.997$ (3 s.f.)
12(iii)	For every unit increase in ln x, the corresponding increase in sales of bottles of sunblock is 1369.
12(iv)	When $x = 10$, $y = –17.326 + 13.698 \ln 10$ $= 14.214$ $\therefore$ Number of bottles sold = 1421 Thus weekly profit = (1421)(20) –10 (2000) = \$8420 Estimate is reliable since $r = 0.997 \approx 1$ and $x = 10$ is within the range of sample data, $8 \le x \le 25$.