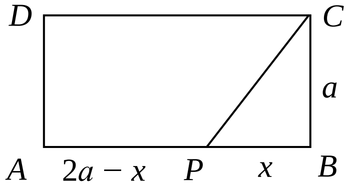
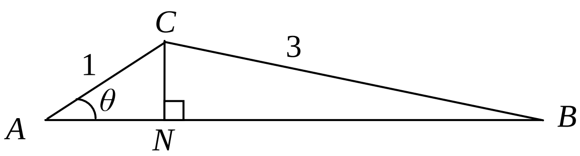
2015 H2 Mathematics Prelim P1 solutions

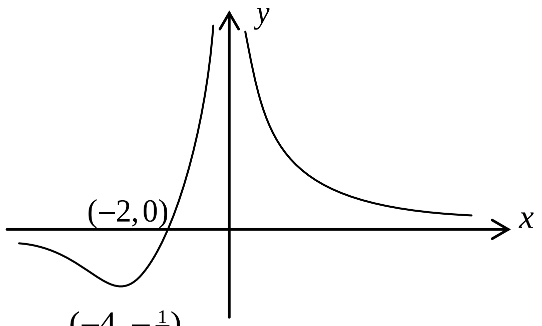
1	$u=3-x^2,\quad x^2=3-u,\quad \frac{\mathrm{d}u}{\mathrm{d}x}=-2x.$ $\int x^3\sqrt{3-x^2}\ \mathrm{d}x=-\frac{1}{2}\int(3-u)u^{\frac{1}{2}}\ \mathrm{d}u$ $=-\frac{1}{2}\int 3u^{\frac{1}{2}}-u^{\frac{3}{2}}\ \mathrm{d}u$ $=\frac{1}{5}u^{\frac{5}{2}}-u^{\frac{3}{2}}+c$ $=\frac{1}{5}(3-x^2)^{\frac{5}{2}}-(3-x^2)^{\frac{3}{2}}+c.$
2	$\frac{x+3}{1-x}\geq\frac{x}{2x+1}$ $\Rightarrow\frac{x}{2x+1}-\frac{x+3}{1-x}\leq 0$ $\Rightarrow\frac{x}{2x+1}+\frac{x+3}{x-1}\leq 0$ $\Rightarrow\frac{x(x-1)+(x+3)(2x+1)}{(2x+1)(x-1)}\leq 0$ $\Rightarrow\frac{(x+1)^2}{(2x+1)(x-1)}\leq 0$ $\therefore -\frac{1}{2}<x<1,\ x=-1$ Replace x by $-\lvert x\rvert$ $\Rightarrow -\frac{1}{2}<-\lvert x\rvert<1, -\lvert x\rvert=-1$ $\Rightarrow -1<\lvert x\rvert<\frac{1}{2}, \lvert x\rvert=1$ $\Rightarrow -\frac{1}{2}<x<\frac{1}{2},\ x=\pm 1$
3	 $T=\frac{2a-x}{100}+\frac{\sqrt{x^2+a^2}}{60}$ $\frac{dT}{dx}=-\frac{1}{100}+\frac{1}{60}\left(\frac{2x}{2\sqrt{x^2+a^2}}\right)$ $\frac{dT}{dx}=0\ \Rightarrow\ -\frac{1}{100}+\frac{x}{60\sqrt{x^2+a^2}}=0$ $\Rightarrow\ \frac{x}{3\sqrt{x^2+a^2}}=\frac{1}{5}$ $\Rightarrow\ 25x^2=9(x^2+a^2)$ $\Rightarrow\ 16x^2=9a^2$ $\Rightarrow\ x=\frac{3}{4}a$ or $x=-\frac{3}{4}a$ (rejected)

	$\frac{d^2T}{dx^2}=\frac{1}{60}\frac{1}{x^2+a^2}\left(\sqrt{x^2+a^2}-\frac{x(2x)}{2\sqrt{x^2+a^2}}\right)$ $=\frac{x^2+a^2-x^2}{60\left(x^2+a^2\right)^{3/2}}$ $=\frac{a^2}{60\left(x^2+a^2\right)^{3/2}}$ When $x=\frac{3}{4}a,\ \frac{d^2T}{dx^2}>0\ \therefore\ T$ is min when $x=\frac{3}{4}a$ min $T=\frac{1}{100}\left(2a-\frac{3}{4}a\right)+\frac{1}{60}\sqrt{\left(\frac{3}{4}a\right)^2+a^2}$ $=\frac{a}{80}+\frac{a}{48}$ $=\frac{a}{30}$ minutes
4(i)	 In $\Delta CNA,\ AN=\cos\theta,\ CN=\sin\theta$ In $\Delta CNB,\ NB=\sqrt{3^2-(CN)^2}=\sqrt{9-\sin^2\theta}$ $\therefore\ AB=AN+NB=\cos\theta+\sqrt{9-\sin^2\theta}$ (shown) **<u>OR</u>** By cosine rule, $3^2=1^2+(AB)^2-2(1)(AB)\cos\theta$ $(AB)^2-2(AB)\cos\theta-8=0$ $AB=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta+32}}{2}$ $AB=\cos\theta+\sqrt{\cos^2\theta+8}$ (since $AB>0$) $AB=\cos\theta+\sqrt{9-\sin^2\theta}$ (shown)
4(ii)	$AB=\cos\theta+\sqrt{9-\sin^2\theta}$ $\approx 1-\frac{1}{2}\theta^2+\left(9-\theta^2\right)^{\frac{1}{2}}$ $=1-\frac{1}{2}\theta^2+9^{\frac{1}{2}}\left(1-\frac{\theta^2}{9}\right)^{\frac{1}{2}}$ $=1-\frac{1}{2}\theta^2+3\left(1-\frac{1}{2}\frac{\theta^2}{9}+...\right)$ $\approx 4-\frac{2}{3}\theta^2$ $\therefore\ a=4,\ b=0,\ c=-\frac{2}{3}$

5	$$\left(1+x^2\right)\frac{dy}{dx}=e^{\tan^{-1}x}$$ $$\left(1+x^2\right)\frac{d^2y}{dx^2}+\left(2x\right)\frac{dy}{dx}=\frac{e^{\tan^{-1}x}}{1+x^2}$$ $$=\frac{dy}{dx}$$ $$\left(1+x^2\right)\frac{d^2y}{dx^2}=\left(1-2x\right)\frac{dy}{dx}\text{ (shown)}$$
	$$\left(1+x^2\right)\frac{d^2y}{dx^2}=\left(1-2x\right)\frac{dy}{dx}$$ $$\left(1+x^2\right)\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}=\left(1-2x\right)\frac{d^2y}{dx^2}-2\frac{dy}{dx}$$ i.e. $$\left(1+x^2\right)\frac{d^3y}{dx^3}=\left(1-4x\right)\frac{d^2y}{dx^2}-2\frac{dy}{dx}$$ When $x=0,\ y=1,\ \frac{dy}{dx}=1,\ \frac{d^2y}{dx^2}=1,\ \frac{d^3y}{dx^3}=-1$ $$\therefore y=1+x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$$ $$=1+x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots$$
	$y=e^{\tan^{-1}x}\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{1+x^2}e^{\tan^{-1}x}\ \Rightarrow\ \left(1+x^2\right)\frac{dy}{dx}=e^{\tan^{-1}x}$ (verified)
	$$y=e^{\tan^{-1}x}=1+x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots$$ $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$ $$\therefore\ e^{\tan^{-1}x}-e^x=\left(1+x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)=-\frac{x^3}{3}$$ $$e^{\tan^{-1}x}=e^x-\frac{x^3}{3}$$ $$\therefore\ k=-\frac{1}{3}$$

6(i)	**<u>Method I (Factor Formula)</u>** $y=0 \Rightarrow x=\frac{\pi}{2}$ or $x=\pi$. $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left\|\cos\frac{x}{2}\cos x\right\| \mathrm{d}x$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\frac{x}{2}\cos x \,\mathrm{d}x-\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos\frac{x}{2}\cos x \,\mathrm{d}x$ $=\frac{1}{2}\left[\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\frac{x}{2}+\cos\frac{3x}{2} \,\mathrm{d}x-\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos\frac{x}{2}+\cos\frac{3x}{2} \,\mathrm{d}x\right]$ $=\frac{1}{2}\left[2\sin\frac{x}{2}+\frac{2}{3}\sin\frac{3x}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}-\frac{1}{2}\left[2\sin\frac{x}{2}+\frac{2}{3}\sin\frac{3x}{2}\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$ $=\frac{1}{2}\left(\sqrt{2}+\frac{\sqrt{2}}{3}-1-\frac{2}{3}\right)-\frac{1}{2}\left(\sqrt{3}+0-\sqrt{2}-\frac{\sqrt{2}}{3}\right)$ $=\left(-\frac{5}{6}+\frac{4}{3}\sqrt{2}-\frac{1}{2}\sqrt{3}\right)$ unit^2. **<u>Method II (Other trigonometric identities)</u>** $y=0 \Rightarrow x=\frac{\pi}{2}$ or $x=\pi$. $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left\|\cos\frac{x}{2}\cos x\right\| \mathrm{d}x$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\frac{x}{2}\left(2\cos^2\frac{x}{2}-1\right)\mathrm{d}x-\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos\frac{x}{2}\left(2\cos^2\frac{x}{2}-1\right)\mathrm{d}x$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\cos^3\frac{x}{2}-\cos\frac{x}{2}\,\mathrm{d}x-\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\cos^3\frac{x}{2}-\cos\frac{x}{2}\,\mathrm{d}x$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\cos\frac{x}{2}\left(1-\sin^2\frac{x}{2}\right)-\cos\frac{x}{2}\,\mathrm{d}x-$ $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\cos\frac{x}{2}\left(1-\sin^2\frac{x}{2}\right)-\cos\frac{x}{2}\,\mathrm{d}x$ $=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\frac{x}{2}-2\cos\frac{x}{2}\sin^2\frac{x}{2}\,\mathrm{d}x-\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos\frac{x}{2}-2\cos\frac{x}{2}\sin^2\frac{x}{2}\,\mathrm{d}x$ $=\left[2\sin\frac{x}{2}-\frac{4}{3}\sin^3\frac{x}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}-\left[2\sin\frac{x}{2}-\frac{4}{3}\sin^3\frac{x}{2}\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$ $=\left(\sqrt{2}-\frac{\sqrt{2}}{3}-1+\frac{1}{6}\right)-\left(\sqrt{3}-\frac{\sqrt{3}}{2}-\sqrt{2}+\frac{\sqrt{2}}{3}\right)$ $=\left(-\frac{5}{6}+\frac{4}{3}\sqrt{2}-\frac{1}{2}\sqrt{3}\right)$ unit^2.
(ii)	$\left\|\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos\frac{x}{2}\cos x \,\mathrm{d}x\right\|$ is the absolute difference between the areas above and below the x-axis, while **(i)** measures the sum of the areas above and below the x-axis.

7(i)	Let P_n be the proposition $u_n = 2(\ln n - n)$ for $n \in \mathbb{Z}^+$. When $n = 1$, LHS $= u_1 = -2$ (given). RHS $= 2(\ln 1 - 1) = -2$. Since LHS = RHS, $\therefore$ P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = 2(\ln k - k)$. We want to show that P_{k+1} is also true, i.e. $u_{k+1} = 2(\ln(k+1) - (k+1)) = 2\ln(k+1) - 2(k+1)$. LHS = $u_{k+1} = u_k + \ln\left(1 + \frac{2k+1}{k^2}\right) - 2$ $= 2(\ln k - k) + \ln\left(1 + \frac{2k+1}{k^2}\right) - 2$ $= 2\ln k - 2k + \ln\left(\frac{k^2+2k+1}{k^2}\right) - 2$ $= 2\ln k - 2k + \ln\left((k+1)^2\right) - \ln\left(k^2\right) - 2$ $= 2\ln(k+1) - 2(k+1)$ = RHS. $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for for all $n \in \mathbb{Z}^+$.
(ii)	$\sum_{r=15}^{n} \frac{e^{u_r}}{r^2} = \sum_{r=15}^{n} \frac{e^{2\ln r - 2r}}{r^2} = \sum_{r=15}^{n} \frac{r^2 e^{-2r}}{r^2} = \sum_{r=15}^{n} e^{-2r}$. $\sum_{r=15}^{n} e^{-2r} = \sum_{r=15}^{n} \left(e^{-2}\right)^r$ $= \frac{e^{-30}\left(1-\left(e^{-2}\right)^{n-14}\right)}{1-e^{-2}}$ $= \frac{1-e^{28-2n}}{e^{30}-e^{28}}$. $= \frac{e^{-28}-e^{-2n}}{e^2-1}$.
(iii)	$\sum_{r=1}^{\infty} \frac{e^{u_r}}{r^2} = \sum_{r=1}^{\infty} e^{-2r}$ converges as $\left\|e^{-2}\right\| < 1$. $\sum_{r=1}^{\infty} e^{-2r} = \frac{e^{-2}}{1-e^{-2}} = \frac{1}{e^2-1}$. (Alternatively, students may use part (ii) by taking n to infinity, and then adding on the sum of the first 14 terms.)

8(i)	$\frac{dx}{dt}=\frac{100-x}{100}$ $\int\frac{1}{100-x}\,dx=\int 0.01\,dt$ $\ln\lvert 100-x\rvert=-0.01t+k$ $100-x=Ae^{-0.01t}, \quad A=\pm e^k$ $x=100-Ae^{-0.01t}$. When $t=0, \quad x=50\Rightarrow A=50.$ $\therefore x=100-50e^{-0.01t}$.
(ii)	As $t\to\infty, \quad x\to 100.$ The population of the predators approaches 100 000 after several years.
(iii)	$\frac{dy}{dt}=-50e^{-0.01t}$. $\int\frac{dy}{dx}\,dx=\int -50e^{-0.01t}\ dt$ $y=5000e^{-0.01t}+c.$ As $t\to\infty, \quad y\to 5000\Rightarrow c=5000$. $\therefore y=5000e^{-0.01t}+5000.$
9(i)	$y=\frac{1}{x}+\frac{2}{x^2}$ <u>Method 1</u> $y=\frac{1}{x}+\frac{2}{x^2}\Rightarrow yx^2=x+2\Rightarrow yx^2-x-2=0$ For range, there must be solutions for *x*, Discriminant $=1-4(y)(-2)\geq 0\Rightarrow y\geq -\frac{1}{8}$ Solution set = $\{y\in\mathbb{R}: y\geq -\frac{1}{8}\}$ <u>Method 2</u> $\frac{dy}{dx}=-\frac{1}{x^2}-\frac{4}{x^3}=0\Rightarrow x=-4$ $\frac{d^2y}{dx^2}=\frac{2}{x^3}+\frac{12}{x^4}$. When $x=-4$, $\frac{d^2y}{dx^2}=\frac{2}{-64}+\frac{12}{256}=\frac{1}{64}>0$ $\therefore(-4,-\frac{1}{8})$ is minimum point Thus Solution set = $\{y\in\mathbb{R}: y\geq -\frac{1}{8}\}$
(ii)	 Asymptotes: $x=0,\ y=0$
	At $x=-1.5,\ y=\frac{2}{9}, x=-1,\ y=1,\ x=1,\ y=3$ Therefore, substituting into the quadratic curve,

$2.25a - 1.5b + c = \frac{2}{9}$

$a - b + c = 1$

$a + b + c = 3$

From GC, $a = -\frac{2}{9}, b = 1, c = \frac{20}{9}$.

Plot the graph $y = -\frac{2}{9}x^2 + x + \frac{20}{9}$.

From GC, $k = 6$

10(a)(i)

Length of projection of $\overrightarrow{OB}$ on $\overrightarrow{OA}$

$$= \left| \overrightarrow{OB} \cdot \frac{\overrightarrow{OA}}{\left|\overrightarrow{OA}\right|} \right|$$

$$= \frac{1}{\sqrt{1+9+9}} \left| \begin{pmatrix} 1 \\ k \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \right|$$

$$= \frac{\left|-1 + 3k + 15\right|}{\sqrt{19}}$$

$$= \frac{\left|14 + 3k\right|}{\sqrt{19}}$$

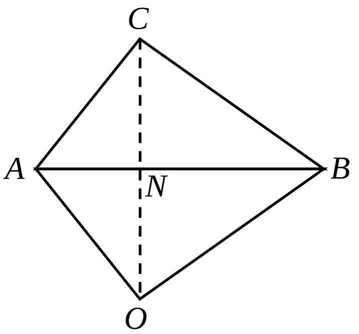
10(a)(ii)

Shortest length of projection is 0, and it occurs when $k = -\frac{14}{3}$.

$\overrightarrow{OA}$ and $\overrightarrow{OB}$ would be perpendicular.

10(b)(i)

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix}$$



Equation of AB is

$$\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

Let N be the foot of perpendicular from O to AB.

Then $\overrightarrow{ON} = \begin{pmatrix} -1+2\lambda \\ 3-5\lambda \\ 3+2\lambda \end{pmatrix}$ for some λ

$\overrightarrow{ON} \perp \overrightarrow{AB} \Rightarrow \begin{pmatrix} -1+2\lambda \\ 3-5\lambda \\ 3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix} = 0$

$\Rightarrow -2 + 4\lambda - 15 + 25\lambda + 6 + 4\lambda = 0$

$\Rightarrow 33\lambda = 11$

$\Rightarrow \lambda = \frac{1}{3}$

$$\overrightarrow{OC} = 2\overrightarrow{ON}$$

$$= 2\begin{pmatrix} -1+\frac{2}{3} \\ 3-\frac{5}{3} \\ 3+\frac{2}{3} \end{pmatrix}$$

$$= \frac{2}{3}\begin{pmatrix} -1 \\ 4 \\ 11 \end{pmatrix} \text{ or } \begin{pmatrix} -2/3 \\ 8/3 \\ 22/3 \end{pmatrix}$$

10(b)(ii)	Area of quadrilateral $= 2\times\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|$ OR $2\times\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{AB}\right\|$ $=\left\|\begin{pmatrix}-1\\3\\3\end{pmatrix}\times\begin{pmatrix}2\\-5\\2\end{pmatrix}\right\|$ $=\left\|\begin{pmatrix}6+15\\-(-2-6)\\5-6\end{pmatrix}\right\|$ $=\sqrt{21^2+8^2+1^2}$ $=\sqrt{506}$
11(a)	Since $\mathrm{P}(x)$ has only real coefficients and p and q are complex roots of $\mathrm{P}(x)=0$, then $p*$ and $q*$ are also complex roots of $\mathrm{P}(x)=0$. Hence least n = 4 $\mathrm{P}(x) = (x-p)(x-p*)(x-q)(x-q*)$ $=[(x-k)-2\mathrm{i}][(x-k)+2\mathrm{i}][(x-3)+3\mathrm{i}][(x-3)-3\mathrm{i}]$ $=[(x-k)^2-(2\mathrm{i})^2][(x-3)^2-(3\mathrm{i})^2]$ $=(x^2-2kx+k^2+4)(x^2-6x+18)$
11(b)(i)	$p=k+2\mathrm{i} \Rightarrow \|p\|=\sqrt{k^2+4}$ $q=3-3\mathrm{i} \Rightarrow \|q\|=\sqrt{3^2+3^2}=3\sqrt{2}$; $\arg(q)=-\tan^{-1}\left(\frac{3}{3}\right)=-\frac{\pi}{4}$ **Method 1:** $\left\|\frac{\mathrm{i}q^2}{2p}\right\|=\frac{\|q\|^2}{2\|p\|}$ $=\frac{(3\sqrt{2})^2}{2\sqrt{k^2+4}}$ $\frac{9}{4}=\frac{9}{\sqrt{k^2+4}}$ **Method 2:** $\frac{\mathrm{i}q^2}{2p}=\frac{\mathrm{i}(3-3\mathrm{i})^2}{2(k+2\mathrm{i})}$ $=\frac{\mathrm{i}(9-18\mathrm{i}-9)}{2(k+2\mathrm{i})}\cdot\frac{(k-2\mathrm{i})}{(k-2\mathrm{i})}$ $=\frac{18(k-2\mathrm{i})}{2(k^2+4)}$ $=\frac{9}{k^2+4}(k-2\mathrm{i})$ $\left\|\frac{\mathrm{i}q^2}{2p}\right\|=\frac{9}{k^2+4}\sqrt{k^2+2^2}$ $\frac{9}{4}=\frac{9}{\sqrt{k^2+4}}$ $16=k^2+4$ $k=\pm\sqrt{16-4}$ $k=2\sqrt{3}$ or $k=-2\sqrt{3}$ (NA since $k>0$)
	Method 1: When $k=2\sqrt{3}$, $\arg(p)=\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)=\frac{\pi}{6}$ **Method 2:** When $k=2\sqrt{3}$, $\alpha=\arg\left(\frac{\mathrm{i}q^2}{2p}\right)$

	$\alpha = \arg\left(\frac{\mathrm{i}q^2}{2p}\right)$ $= \arg(\mathrm{i}) + \arg(q^2) - \arg(2p)$ $= \frac{\pi}{2} + 2\left(-\frac{\pi}{4}\right) - \frac{\pi}{6}$ $= -\frac{\pi}{6}$	$= -\tan^{-1}\left(\frac{2}{2\sqrt{3}}\right)$ $= -\frac{\pi}{6}$
11(b)(ii)	$\mathrm{i}z^4 = -\frac{q^2}{2p}$ $z^4 = -\frac{q^2}{2p}\left(\frac{1}{\mathrm{i}}\right)\left(\frac{\mathrm{i}}{\mathrm{i}}\right)$ $z^4 = \frac{\mathrm{i}q^2}{2p}$ $= \frac{9}{4}\mathrm{e}^{\left(-\frac{\pi}{6}+2k\pi\right)\mathrm{i}}$, $k = 0, \ \pm 1, \ 2$ $z = \left(\frac{9}{4}\right)^{\frac{1}{4}}\mathrm{e}^{\left(-\frac{\pi}{24}+\frac{k\pi}{2}\right)\mathrm{i}}$, $k = 0, \ \pm 1, \ 2$ $= \sqrt{\frac{3}{2}}\mathrm{e}^{\left(-\frac{13\pi}{24}\right)\mathrm{i}}, \ \sqrt{\frac{3}{2}}\mathrm{e}^{\left(-\frac{\pi}{24}\right)\mathrm{i}}, \ \sqrt{\frac{3}{2}}\mathrm{e}^{\left(\frac{11\pi}{24}\right)\mathrm{i}}, \ \sqrt{\frac{3}{2}}\mathrm{e}^{\left(\frac{23\pi}{24}\right)\mathrm{i}}$	
12(i)	$x = \frac{u}{1+u} \ \Rightarrow \ \frac{dx}{du} = \frac{(1+u)-u}{(1+u)^2} = \frac{1}{(1+u)^2}$ $y = \frac{u^2}{1+u} \ \Rightarrow \ \frac{dy}{du} = \frac{(1+u)(2u)-u^2}{(1+u)^2} = \frac{u^2+2u}{(1+u)^2}$ $\therefore \ \frac{dy}{dx} = \frac{u^2+2u}{(1+u)^2} \div \frac{1}{(1+u)^2} = u^2 + 2u$	
12(ii)	$\frac{du}{dt} = 2, \ u = 1$ $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dt}$ $= (2u+2)(2)$ $= 8$ units per second	
12(iii)	When $x = \frac{1}{2}$ (i.e. $u = 1$), $\frac{dy}{dx} = 3$ Gradient of tangent at $x = \frac{1}{2}$ is 3 When $u = -\frac{1}{2}$, $\frac{dy}{dx} = \frac{1}{4} - 1 = -\frac{3}{4}$ Gradient of normal at $u = -\frac{1}{2}$ is $\frac{4}{3}$ $\therefore$ acute angle between them is $\tan^{-1} 3 - \tan^{-1}\frac{4}{3}$ $= 71.57^\circ - 53.13^\circ = 18.4^\circ$	

12(iv)	Equation of tangent at P is $y-\frac{p^2}{1+p}=\left(p^2+2p\right)\left(x-\frac{p}{1+p}\right)$ At y-axis, when $x = 0$, $y=\frac{p^2}{1+p}-\frac{\left(p^2+2p\right)p}{1+p}=\frac{p^2\left(-1-p\right)}{1+p}=-p^2$ $\therefore$ Tangent at P cuts the y-axis at $\left(0,-p^2\right)$ Similarly, tangent at Q cuts the y-axis at $\left(0,-q^2\right)$ $-p^2=-q^2 \Rightarrow p=\pm q$ Since $p\neq q,\ p=-q \Rightarrow p+q=0$ (shown)
13(i)	To find least n such that $T_n=15+\left(n-1\right)\left(0.5\right)>39$. $0.5n>24.5$ $n>49$. Therefore, on the 50th day of rental the owner will first have to pay the artist more than \$39 as the daily rental rate.
13(ii)	$\mathrm{f}\left(n\right)=\frac{12000}{4n^2+4n-3}$. $\mathrm{f}\left(n\right)=\frac{12000}{\left(2n-1\right)\left(2n+3\right)}$ $=\frac{3000}{2n-1}-\frac{3000}{2n+3}$.
13(iii)	$\sum_{r=1}^{r=m}\mathrm{f}\left(\mathrm{r}\right)=3000\sum_{r=1}^{r=m}\frac{1}{2r-1}-\frac{1}{2r+3}$ $=3000\begin{pmatrix}\frac{1}{1} & -\frac{1}{5}\\ +\frac{1}{3} & -\frac{1}{7}\\ +\frac{1}{5} & -\frac{1}{9}\\ \vdots & \\ +\frac{1}{2m-5} & -\frac{1}{2m-1}\\ +\frac{1}{2m-3} & -\frac{1}{2m+1}\\ +\frac{1}{2m-1} & -\frac{1}{2m+3}\end{pmatrix}$ $=3000\left(1+\frac{1}{3}-\frac{1}{2m+1}-\frac{1}{2m+3}\right)$ $=3000\left(\frac{4}{3}-\frac{4m+4}{\left(2m+1\right)\left(2m+3\right)}\right)$ $=4000-\frac{12000\left(m+1\right)}{\left(2m+1\right)\left(2m+3\right)}$.
13(iv)	Given:

$$\frac{k}{2}\left[2(15)+(k-1)(0.5)\right]<4000-\frac{12000(k+1)}{(2k+1)(2k+3)}.$$

Considering

$$Y_1=\frac{k}{2}\left[2(15)+(k-1)(0.5)\right]-4000+\frac{12000(k+1)}{(2k+1)(2k+3)},$$

k	Y_1
98	-123.2
99	-59.5
100	4.7037

By GC, largest value of k is 99.